Existence and uniqueness of E_∞-structures on motivic K-theory spectra

Niko Naumann, Markus Spitzweck, Paul Arne Østvær

June 30, 2011

Abstract

The algebraic K-theory spectrum KGL, the motivic Adams summand ML and their connective covers have unique E_∞-structures refining their naive multiplicative structures in the motivic stable homotopy category. These results are deduced from Γ-homology computations in motivic obstruction theory.

1 Introduction

Motivic homotopy theory intertwines classical algebraic geometry and stable algebraic topology. In this paper we study obstruction theory for E_∞-structures in the motivic setup. An E_∞-structure on a spectrum refers as usual to a ring structure which is not just given up to homotopy, but where the homotopies encode a coherent homotopy commutative multiplication. Many of the examples of motivic ring spectra begin life as commutative monoids in the motivic stable homotopy category. We are interested in the following questions: When can the multiplicative structure of a given commutative monoid in the motivic stable homotopy category be refined to an E_∞-ring spectrum? And if such a refinement exists, is it unique? The questions of existence and uniqueness of E_∞-structures and their many ramifications have been studied extensively in topology. The first motivic examples worked out in this paper are of K-theoretic interest.

The complex cobordism spectrum MU and its motivic analogue MGL have natural E_∞-structures. In the topological setup, Baker and Richter [1] have shown that the complex K-theory spectrum KU, the Adams summand L and the real K-theory spectrum KO admit unique E_∞-structures. The results in [1] are approached via the obstruction theory developed by Robinson in [11], where it is shown that existence and uniqueness of E_∞-structures are guaranteed provided certain Γ-cohomology groups vanish.

In our approach we rely on analogous results in the motivic setup, see [12] for a further generalization. We show that the relevant motivic Γ-cohomology groups vanish in the case of the algebraic K-theory spectrum KGL (Theorem 2.6) and the motivic Adams summand ML (see §4). The main ingredients in the proofs are new computations of the Γ-homology complexes of KU and L, see Theorem 2.3 and Lemma 4.3, and the Landweber base change formula for the motivic cooperations of KGL and ML. Our main result for KGL can be formulated as follows:

Theorem: *The algebraic K-theory spectrum KGL has a unique E_∞-structure refining its multiplication in the motivic stable homotopy category.*

The existence of the E_∞-structure on KGL was already known using the Bott inverted model for algebraic K-theory, see [13], [15], [2], but the analogous result for ML is new. The uniqueness part of the Theorem is new, and it rules out the existence of any exotic E_∞-structures on KGL. We note that related motivic E_∞-structures have proven useful in the recent constructions of Atiyah-Hirzebruch types of spectral sequences for motivic twisted K-theory [14].

One may ask if the uniqueness of E_∞-structures on KGL has any consequences for the individual algebraic K-theory spectra of smooth schemes over a fixed base scheme. If the base scheme is regular, consider the following presheaves of E_∞-ring spectra. The first one arises from evaluating the E_∞-spectrum KGL on individual smooth schemes, and the second one from a functorial construction of algebraic K-theory spectra, cf. [7]. It is natural to ask if these two presheaves are equivalent in some sense. If the second presheaf is obtained from a motivic E_∞-spectrum, then our uniqueness result would answer this question in the affirmative. The K-theory presheaf has this property when viewed as an A_∞-object, see [8], but as an E_∞-object this is still an open problem.

In topology, the Goerss-Hopkins-Miller obstruction theory [3] allows to gain control over moduli spaces of E_∞-structures. In favorable cases, such as for Lubin-Tate spectra, the moduli spaces are $K(\pi, 1)$'s giving rise to actions of certain automorphism groups as E_∞-maps. A motivic analogue of this obstruction theory has not been worked out. One reason for doing so is that having a homotopy ring structure on a spectrum is often not sufficient in order to form homotopy fixed points under a group action. In Subsection 2.3 we note an interesting consequence concerning E_∞-structures on hermitian K-theory.

In Section 3 we show that the connective cover kgl of the algebraic K-theory spectrum has a unique E_∞-structure, and ditto in Section 4 for the connective cover of the Adams summand.

2 Algebraic K-theory KGL

In this section we shall present the Γ-cohomology computation showing there is a unique E_∞-structure on the algebraic K-theory spectrum KGL. Throughout we work over some noetherian base scheme of finite Krull dimension, which we omit from the notation.

There are two main ingredients which make this computation possible: First, the Γ-homology computation of $\mathsf{KU}_0\mathsf{KU}$ over $\mathsf{KU}_0 = \mathbf{Z}$, where KU is the complex K-theory spectrum. Second, we employ base change for the motivic cooperations of algebraic K-theory, as shown in our previous work [9].

2.1 The Γ-homology of $\mathsf{KU}_0\mathsf{KU}$ over KU_0

For a map $A \to B$ between commutative algebras we denote Robinson's Γ-homology complex by $\widetilde{\mathcal{K}}(B|A)$ [11, Definition 4.1]. Recall that $\widetilde{\mathcal{K}}(B|A)$ is a homological double complex of B-modules concentrated in the first quadrant. The same construction can be performed for maps between graded and bigraded algebras. In all cases we let $\mathcal{K}(B|A)$ denote the total complex associated with the double complex $\widetilde{\mathcal{K}}(B|A)$.

The Γ-cohomology

$$\mathsf{H}\Gamma^*(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0, -) = \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathsf{KU}_0\mathsf{KU}}(\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0), -)$$

has been computed for various coefficients in [1]. In what follows we require precise information about the complex $\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0)$, since it satisfies a motivic base change property, cf. Lemma 2.4.

Lemma 2.1: *Let $X \in \mathsf{Ch}_{\geq 0}(\mathsf{Ab})$ be a non-negative chain complex of abelian groups. The following are equivalent:*

i) The canonical map $X \longrightarrow X \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{Q} = X \otimes_{\mathbf{Z}} \mathbf{Q}$ is a quasi isomorphism.

ii) For every prime p, there is a quasi isomorphism $X \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p \simeq 0$.

Proof. It is well known that X is formal [4, pg. 164], i.e. there is a quasi isomorphism

$$X \simeq \bigoplus_{n\geq 0} H_n(X)[n].$$

(For an abelian group A and integer n, we let $A[n]$ denote the chain complex that consists of A concentrated in degree n.) Hence for every prime p,

$$X \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p \simeq \bigoplus_{n\geq 0} \left(H_n(X)[n] \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p\right).$$

By resolving $\mathbf{F}_p = (\mathbf{Z} \xrightarrow{\cdot p} \mathbf{Z})$ one finds an isomorphism

$$H_*(A[n] \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p) \cong (A/pA)[n] \oplus A\{p\}[n+1]$$

for every abelian group A and integer n. Here $A\{p\}$ is shorthand for $\{x \in A \,|\, px = 0\}$. In summary, ii) holds if and only if the multiplication by p map

$$\cdot p : H_*(X) \longrightarrow H_*(X)$$

is an isomorphism for every prime p. The latter is equivalent to i). $\square$

We shall use the previous lemma in order to study cotangent complexes introduced by Illusie in [6]. Let R be a ring and set $R_{\mathbf{Q}} := R \otimes_{\mathbf{Z}} \mathbf{Q}$. Then there is a canonical map

$$\tau_R : \mathbb{L}_{R/\mathbf{Z}} \longrightarrow \mathbb{L}_{R/\mathbf{Z}} \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{Q} \simeq \mathbb{L}_{R/\mathbf{Z}} \otimes^{\mathbf{L}}_{R} R_{\mathbf{Q}} \xrightarrow{\simeq} \mathbb{L}_{R_{\mathbf{Q}}/\mathbf{Q}}$$

of cotangent complexes in $\mathsf{Ho}(\mathsf{Ch}_{\geq 0}(\mathbf{Z}))$. The first quasi isomorphism is obvious, while the second one is an instance of flat base change for cotangent complexes.

Lemma 2.2: *The following are equivalent:*

i) τ_R is a quasi isomorphism.

ii) For every prime p, there is a quasi isomorphism $\mathbb{L}_{R/\mathbf{Z}} \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p \simeq 0$.

If the abelian group underlying R is torsion free, then i) and ii) are equivalent to

iii) For every prime p, $\mathbb{L}_{(R/pR)/\mathbf{F}_p} \simeq 0$.

Proof. The equivalence of i) and ii) follows by applying Lemma 2.1 to $X = \mathbb{L}_{R/\mathbf{Z}}$. If R is torsion free, then it is flat as a $\mathbf{Z}$-algebra. Hence, by flat base change, there exists a quasi isomorphism

$$\mathbb{L}_{R/\mathbf{Z}} \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p \simeq \mathbb{L}_{(R/pR)/\mathbf{F}_p}.$$

$\square$

The following is our analogue for Robinson's Γ-homology complex of the Baker-Richter result [1, Theorem 5.1].

Theorem 2.3: *i) Let R be a torsion free ring such that $\mathbb{L}_{(R/pR)/\mathbf{F}_p} \simeq 0$ for every prime p, e.g. assume that $\mathbf{F}_p \to R/pR$ is ind-étale for all p. Then there is a quasi isomorphism*

$$\mathcal{K}(R|\mathbf{Z}) \simeq \mathcal{K}(R_{\mathbf{Q}}|\mathbf{Q})$$

in the derived category of R-modules.

ii) There is a quasi isomorphism

$$\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0) \simeq (\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}}[0]$$

in the derived category of $\mathsf{KU}_0\mathsf{KU}$*-modules.*

Proof. i) The Atiyah-Hirzebruch spectral sequence noted in [10, Remark 2.3] takes the form

$$E^2_{p,q} = H^p(\mathbb{L}_{R/\mathbf{Z}} \otimes^{\mathbf{L}}_{\mathbf{Z}} \Gamma^q(\mathbf{Z}[x]|\mathbf{Z})) \Rightarrow H^{p+q}(\mathcal{K}(R|\mathbf{Z})).$$

Our assumptions on R and Lemma 2.2 imply that the E^2-page is comprised of $\mathbf{Q}$-vector spaces. Hence so is the abutment, and there exists a quasi isomorphism between complexes of R-modules

$$\mathcal{K}(R|\mathbf{Z}) \xrightarrow{\simeq} \mathcal{K}(R|\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Moreover, by Lemma 2.4, there is a quasi isomorphism

$$\mathcal{K}(R|\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \mathcal{K}(R_{\mathbf{Q}}|\mathbf{Q}).$$

ii) According to [1, Theorem 3.1, Corollary 3.4, (a)] and the Hopf algebra isomorphism $A^{st} \simeq \mathsf{KU}_0\mathsf{KU}$ [1, Proposition 6.1], the ring $R := \mathsf{KU}_0\mathsf{KU}$ satisfies the assumptions of part i)[1]. Now since $\mathsf{KU}_0 \cong \mathbf{Z}$,

$$\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0) \simeq \mathcal{K}((\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}}|\mathbf{Q}).$$

We have that $(\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}} \simeq \mathbf{Q}[w^{\pm 1}]$ [1, Theorem 3.2, (c)] is a smooth $\mathbf{Q}$-algebra. Hence, since Γ-cohomology agrees with André-Quillen cohomology over $\mathbf{Q}$, there are quasi isomorphisms

$$\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0) \simeq \Omega^1_{\mathbf{Q}[w^{\pm 1}]|\mathbf{Q}}[0] \simeq (\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}}[0].$$

□

2.2 The Γ-homology of $\mathsf{KGL}_{**}\mathsf{KGL}$ over KGL_{**}

The strategy in what follows is to combine the computations for KU in §2.1 with motivic Landweber exactness [9]. To this end we require the following general base change result, which was also used in the proof of Theorem 2.3.

[1] This follows also easily from Landweber exactness of KU.

Lemma 2.4: *For a pushout of ordinary, graded or bigraded commutative algebras*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

there are isomorphisms between complexes of D-modules

$$\mathcal{K}(D|C) \cong \mathcal{K}(B|A) \otimes_B D \cong \mathcal{K}(B|A) \otimes_A C.$$

If B is flat over A, then $\widetilde{\mathcal{K}}(B|A)$ is a first quadrant homological double complex of flat B-modules; thus, in the derived category of D-modules there are quasi isomorphisms

$$\mathcal{K}(D|C) \simeq \mathcal{K}(B|A) \otimes^{\mathbf{L}}_B D \simeq \mathcal{K}(B|A) \otimes^{\mathbf{L}}_A C.$$

Proof. Following the notation in [11, §4], let $(B|A)^{\otimes}$ denote the tensor algebra of B over A. Then $(B|A)^{\otimes} \otimes_A B$ has a natural Γ-module structure over B, cf. [11, §4]. Here Γ denotes the category of finite based sets and basepoint preserving maps. It follows that $((B|A)^{\otimes} \otimes_A B) \otimes_B D$ is a Γ-module over D. Moreover, by base change for tensor algebras, there exists an isomorphism of Γ-modules in D-modules

$$((B|A)^{\otimes} \otimes_A B) \otimes_B D \cong (D|C)^{\otimes} \otimes_C D.$$

Here we use that the Γ-module structure on $(B|A)^{\otimes} \otimes_A M$, for M a B-module, is given as follows: For a map $\varphi\colon [m] \to [n]$ between finite pointed sets,

$$(B \otimes_A B \otimes_A \cdots \otimes_A B) \otimes_A M \to (B \otimes_A B \otimes_A \cdots \otimes_A B) \otimes_A M$$

sends $b_1 \otimes \cdots \otimes b_m \otimes m$ to

$$\Big(\prod_{i\in\varphi^{-1}(1)} b_i\Big) \otimes \cdots \otimes \Big(\prod_{i\in\varphi^{-1}(n)} b_i\Big) \otimes \Big(\Big(\prod_{i\in\varphi^{-1}(0)} b_i\Big) \cdot m\Big).$$

By convention, if $\varphi^{-1}(j) = \emptyset$ then $\prod_{i\in\varphi^{-1}(j)} b_i = 1$. Robinson's Ξ-construction yields an isomorphism between double complexes of D-modules

$$\widetilde{\mathcal{K}}(D|C) = \Xi((D|C)^{\otimes} \otimes_C D) \cong \Xi(((B|A)^{\otimes} \otimes_A B) \otimes_B D).$$

Inspection of the Ξ-construction reveals there is an isomorphism

$$\Xi(((B|A)^{\otimes} \otimes_A B) \otimes_B D) \cong \Xi((B|A)^{\otimes} \otimes_A B) \otimes_B D.$$

By definition, this double complex of D-modules is $\widetilde{\mathcal{K}}(B|A) \otimes_B D \cong \widetilde{\mathcal{K}}(B|A) \otimes_A C$. This proves the first assertion by comparing the corresponding total complexes. The remaining claims follow easily. $\square$

Next we recall the structure of the motivic cooperations of the algebraic K-theory spectrum KGL. The algebras we shall consider are bigraded as follows: $\mathsf{KU}_0 \cong \mathbf{Z}$ in bidegree $(0,0)$ and $\mathsf{KU}_* \cong \mathbf{Z}[\beta^{\pm 1}]$ with the Bott-element β in bidegree $(2,1)$. With these conventions, there is a canonical bigraded map

$$\mathsf{KU}_* \to \mathsf{KGL}_{**}.$$

Lemma 2.5: *There are pushouts of bigraded algebras*

$$\begin{tikzcd} \mathsf{KU}_* \arrow[r, "\eta_L"] \arrow[d] & \mathsf{KU}_*\mathsf{KU} \arrow[d] \\ \mathsf{KGL}_{**} \arrow[r, "\eta_L"] & \mathsf{KGL}_{**}\mathsf{KGL} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{KU}_0 \arrow[r, "(\eta_L)_0"] \arrow[d] & \mathsf{KU}_0\mathsf{KU} \arrow[d] \\ \mathsf{KU}_* \arrow[r, "\eta_L"] & \mathsf{KU}_*\mathsf{KU} \end{tikzcd}$$

and a quasi isomorphism in the derived category of $\mathsf{KGL}_{**}\mathsf{KGL}$*-modules*

$$\mathcal{K}(\mathsf{KGL}_{**}\mathsf{KGL}|\mathsf{KGL}_{**}) \simeq \mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0) \otimes^{\mathbf{L}}_{\mathsf{KU}_0\mathsf{KU}} \mathsf{KGL}_{**}\mathsf{KGL}.$$

Proof. Here, η_L is a generic notation for the left unit of some flat Hopf-algebroid. The first pushout is shown in [9, Proposition 9.1, (c)]. The second pushout is in [1]. Applying Lemma 2.4 twice gives the claimed quasi isomorphism. $\square$

Next we compute the Γ-cohomology of the motivic cooperations of KGL.

Theorem 2.6: *i) There is an isomorphism*

$$\mathsf{H}\Gamma^{*,*,*}(\mathsf{KGL}_{**}\mathsf{KGL}|\mathsf{KGL}_{**};\mathsf{KGL}_{**}) \cong \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathbf{Z}}(\mathbf{Q}[0],\mathsf{KGL}_{**}).$$

ii) For all $s \geq 2$,

$$\mathsf{H}\Gamma^{s,*,*}(\mathsf{KGL}_{**}\mathsf{KGL}|\mathsf{KGL}_{**};\mathsf{KGL}_{**}) = 0.$$

Proof. i) By the definition of Γ-cohomology and the results in this Subsection there are isomorphisms

$$\begin{aligned} & \mathsf{H}\Gamma^{*,*,*}(\mathsf{KGL}_{**}\mathsf{KGL}|\mathsf{KGL}_{**};\mathsf{KGL}_{**}) \\ = \; & \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathsf{KGL}_{**}\mathsf{KGL}}(\mathcal{K}(\mathsf{KGL}_{**}\mathsf{KGL}|\mathsf{KGL}_{**}),\mathsf{KGL}_{**}) \\ \cong \; & \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathsf{KGL}_{**}\mathsf{KGL}}(\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathbf{Z}) \otimes^{\mathbf{L}}_{\mathsf{KU}_0\mathsf{KU}} \mathsf{KGL}_{**}\mathsf{KGL},\mathsf{KGL}_{**}) \\ \cong \; & \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathsf{KU}_0\mathsf{KU}}(\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathbf{Z}),\mathsf{KGL}_{**}) \\ \cong \; & \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathsf{KU}_0\mathsf{KU}}((\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}}[0],\mathsf{KGL}_{**}) \\ \cong \; & \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathbf{Z}}(\mathbf{Q}[0],\mathsf{KGL}_{**}). \end{aligned}$$

ii) This follows from i) since $\mathbf{Z}$ has global dimension 1.

$\square$

Remark 2.7: *It is an exercise to compute* $\mathbf{R}\mathsf{Hom}_{\mathbf{Z}}(\mathbf{Q}, -)$ *for finitely generated abelian groups. This explicates our Gamma-cohomology computation in degrees* 0 *and* 1 *for base schemes with finitely generated algebraic* K*-groups, e.g. finite fields and number rings. The computation* $\mathbf{R}\mathsf{Hom}_{\mathbf{Z}}(\mathbf{Q}, \mathbf{Z}) \simeq \hat{\mathbf{Z}}/\mathbf{Z}[1]$ *shows our results imply [1, Corollary 5.2].*

The vanishing result Theorem 2.6, ii) together with the motivic analogues of the results in [11, Theorem 5.6], as detailed in [12], conclude the proof of the Theorem for KGL formulated in the Introduction.

2.3 A remark on hermitian K-theory KQ

In this short Subsection we discuss one instance in which the motivic obstruction theory used here falls short of a putative motivic analogue of the obstruction theory of Goerss, Hopkins and Miller [3]. By [9, Theorem 9.7, (ii), Remark 9.8, (iii)] we may realize the stable Adams operation Ψ^{-1} on algebraic K-theory by a motivic ring spectrum map

$$\Psi^{-1} : \mathsf{KGL} \longrightarrow \mathsf{KGL}. \tag{1}$$

In many cases of interest one expects that $\mathsf{fib}(\psi^{-1} - 1)$ represents Hermitian K-theory KQ. A motivic version of the Goerss-Hopkins-Miller obstruction theory in [3] implies, in combination with Theorem 2.6, that (1) can be modelled as an E_∞-map. With this result in hand, it would follow that KQ admits an E_∞-structure.

It seems the obstruction theory we use is intrinsically unable to provide such results by "computing" E_∞-mapping spaces. However, there might be a more direct way of showing that KQ has a unique E_∞-structure, using the obstruction theory in this paper. A first step would be to compute the motivic cooperations of KQ.

3 Connective algebraic K-theory kgl

We define the connective algebraic K-theory spectrum kgl as the effective part $\mathsf{f}_0\mathsf{KGL}$ of KGL. Recall that the functor f_i defined in [16] projects from the motivic stable homotopy category to its ith effective part. Note that $\mathsf{f}_0\mathsf{KGL}$ is a commutative monoid in the motivic stable homotopy category since projection to the effective part is a lax symmetric monoidal functor (because it is right adjoint to a monoidal functor). For $i \in \mathbf{Z}$ there exists a natural map $\mathsf{f}_{i+1}\mathsf{KGL} \to \mathsf{f}_i\mathsf{KGL}$ in the motivic stable homotopy category with

cofiber the ith slice of KGL. With these definitions, $\mathsf{KGL} \cong \operatorname{hocolim} \mathsf{f}_i\mathsf{KGL}$ (this is true for any motivic spectrum, cf. [16, Lemma 4.2]). Bott periodicity for algebraic K-theory implies that $\mathsf{f}_{i+1}\mathsf{KGL} \cong \Sigma^{2,1}\mathsf{f}_i\mathsf{KGL}$. This allows to recast the colimit as $\operatorname{hocolim} \Sigma^{2i,i}\mathsf{kgl}$ with multiplication by the Bott element β in $\mathsf{kgl}^{-2,-1} \cong \mathsf{KGL}^{-2,-1}$ as the transition map at each stage. We summarize these observations in a lemma.

Lemma 3.1: *The algebraic K-theory spectrum KGL is isomorphic in the motivic stable homotopy category to the Bott inverted connective algebraic K-theory spectrum $\mathsf{kgl}[\beta^{-1}]$.*

Theorem 3.2: *The connective algebraic K-theory spectrum kgl has a unique E_∞-structure refining its multiplication in the motivic stable homotopy category.*

Proof. The connective cover functor f_0 preserves E_∞-structures [5]. Thus the existence of an E_∞-structure on kgl is ensured. We note that inverting the Bott element can be refined to the level of motivic E_∞-ring spectra by the methods employed in [13]. Thus, by Lemma 3.1, starting out with any two E_∞-structures on kgl produces two E_∞-structures on KGL, which coincide by the uniqueness result for E_∞-structures on KGL . Applying f_0 recovers the two given E_∞-structures on kgl: If X is E_∞ with $\varphi\colon X \simeq \mathsf{kgl}$ as ring spectra, then there is a canonical E_∞-map $X \to X[\beta'^{-1}]$, where β' is the image of the Bott element under φ. Since X is an effective motivic spectrum, this map factors as an E_∞-map $X \to \mathsf{f}_0(X[\beta'^{-1}])$. By construction of kgl the latter map is an equivalence. This shows the two given E_∞-structures on kgl coincide. $\square$

4 The motivic Adams summands ML and ml

Let BP denote the Brown-Peterson spectrum for a fixed prime number p. Then the coefficient ring $\mathsf{KU}_{(p)*}$ of the p-localized complex K-theory spectrum is a BP_*-module via the ring map $\mathsf{BP}_* \to \mathsf{MU}_{(p)*}$ which classifies the p-typicalization of the formal group law over $\mathsf{MU}_{(p)*}$. The $\mathsf{MU}_{(p)*}$-algebra structure on $\mathsf{KU}_{(p)*}$ is induced from the natural orientation $\mathsf{MU} \to \mathsf{KU}$. With this BP_*-module structure, $\mathsf{KU}_{(p)*}$ splits into a direct sum of the $\Sigma^{2i}\mathsf{L}_*$ for $0 \leq i \leq p-2$, where L is the Adams summand of $\mathsf{KU}_{(p)}$. Thus motivic Landweber exactness [9] over the motivic Brown-Peterson spectrum MBP produces a splitting of motivic spectra

$$\mathsf{KGL}_{(p)} = \bigvee_{i=0}^{p-2} \Sigma^{2i,i}\mathsf{ML}.$$

We refer to ML as the motivic Adams summand of algebraic K-theory.

Since L_* is an BP_*-algebra and there are no nontrivial phantom maps from any smash power of ML to ML, which follows from [9, Remark 9.8, (ii)] since ML is a retract of $\mathsf{KGL}_{(p)}$, we deduce that the corresponding ring homology theory induces a commutative monoid structure on ML in the motivic stable homotopy category.

We define the connective motivic Adams summand ml to be $f_0\mathsf{ML}$. It is also a commutative monoid in the motivic homotopy category.

Theorem 4.1: *The motivic Adams summand* ML *has a unique* E_∞*-structure refining its multiplication in the motivic stable homotopy category. The same result holds for the connective motivic Adams summand* ml*.*

The construction of ML as a motivic Landweber exact spectrum makes the following result evident on account of the proof of Lemma 2.5.

Lemma 4.2: *There exist pushout squares of bigraded algebras*

$$\begin{tikzcd} \mathsf{L}_* \arrow[r, "\eta_L"] \arrow[d] & \mathsf{L}_*\mathsf{L} \arrow[d] \\ \mathsf{ML}_{**} \arrow[r, "\eta_L"] & \mathsf{ML}_{**}\mathsf{ML} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{L}_0 \arrow[r, "(\eta_L)_0"] \arrow[d] & \mathsf{L}_0\mathsf{L} \arrow[d] \\ \mathsf{L}_* \arrow[r, "\eta_L"] & \mathsf{L}_*\mathsf{L} \end{tikzcd}$$

and a quasi isomorphism in the derived category of $\mathsf{ML}_{**}\mathsf{ML}$*-modules*

$$\mathcal{K}(\mathsf{ML}_{**}\mathsf{ML}|\mathsf{ML}_{**}) \simeq \mathcal{K}(\mathsf{L}_0\mathsf{L}|\mathsf{L}_0) \otimes^{\mathbf{L}}_{\mathsf{L}_0\mathsf{L}} \mathsf{ML}_{**}\mathsf{ML}.$$

Next we show the analog of Theorem 2.3, ii) for the motivic Adams summand.

Lemma 4.3: *In the derived category of* $\mathsf{L}_0\mathsf{L}$*-modules, there is a quasi isomorphism*

$$\mathcal{K}(\mathsf{L}_0\mathsf{L}|\mathsf{L}_0) \simeq (\mathsf{L}_0\mathsf{L})_{\mathbf{Q}}[0].$$

Proof. In the notation of [1, Proposition 6.1] there is an isomorphism between Hopf algebras $\mathsf{L}_0\mathsf{L} \cong {}^{\zeta}A^{st}_{(p)}$. Recall that ${}^{\zeta}A^{st}_{(p)}$ is a free $\mathbf{Z}_{(p)}$-module on a countable basis and ${}^{\zeta}A^{st}_{(p)}/p{}^{\zeta}A^{st}_{(p)}$ is a formally étale $\mathbf{F}_p$-algebra [1, Theorem 3.3(c), Corollary 4.2]. Applying Theorem 2.3, i) to $R = \mathsf{L}_0\mathsf{L}$ and using that $(\mathsf{L}_0\mathsf{L})_{\mathbf{Q}} \simeq \mathbf{Q}[v^{\pm 1}]$ by Landweber exactness, where $v = w^{p-1}$ and $(\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}} \cong \mathbf{Q}[w^{\pm 1}]$, we find

$$\mathcal{K}(\mathsf{L}_0\mathsf{L}|\mathsf{L}_0) \simeq \Omega^1_{\mathbf{Q}[v^{\pm 1}]|\mathbf{Q}}[0] \simeq (\mathsf{L}_0\mathsf{L})_{\mathbf{Q}}[0].$$

$\square$

Lemmas 4.2 and 4.3 imply there is a quasi isomorphism

$$\mathsf{H}\Gamma^{*,*,*}(\mathsf{ML}_{**}\mathsf{ML}|\mathsf{ML}_{**};\mathsf{ML}_{**}) \simeq \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathbf{Z}}(\mathbf{Q}[0],\mathsf{ML}_{**}).$$

Thus the part of Theorem 4.1 dealing with ML follows, since for all $s \geq 2$,

$$\mathsf{H}\Gamma^{s,*,*}(\mathsf{ML}_{**}\mathsf{ML}|\mathsf{ML}_{**};\mathsf{ML}_{**}) = 0. \tag{2}$$

The assertion about ml follows by the exact same type of argument as for kgl. The periodicity operator in this case is $v_1 \in \mathsf{ml}^{2(1-p),1-p} = \mathsf{ML}^{2(1-p),1-p}$.

Acknowledgements. The main result of this paper was announced by the first named author at the 2009 Münster workshop on Motivic Homotopy Theory. He thanks the organizers E. M. Friedlander, G. Quick and P. A. Østvær for the invitation, and P. A. Østvær for hospitality while visiting the University of Oslo, where the major part of this work was finalized.

References

[1] A. Baker and B. Richter. On the Γ-cohomology of rings of numerical polynomials and E_∞ structures on K-theory. *Comment. Math. Helv.*, 80(4):691–723, 2005.

[2] D. Gepner and V. Snaith. On the motivic spectra representing algebraic cobordism and algebraic K-theory. *Doc. Math.*, 14:359–396, 2009.

[3] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004.

[4] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[5] J. J. Gutiérrez, O. Röndigs, M. Spitzweck, and P. A. Østvær. Slices and colored operads. In preparation.

[6] L. Illusie. *Complexe cotangent et déformations. I.* Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin, 1971.

[7] J. F. Jardine. The K-theory presheaf of spectra. In *New topological contexts for Galois theory and algebraic geometry (BIRS 2008)*, volume 16 of *Geom. Topol. Monogr.*, pages 151–178. Geom. Topol. Publ., Coventry, 2009.

[8] Y. Kim. *Motivic Symmetric Ring Spectrum Representing Algebraic K-theory.* PhD thesis, University of Illinois at Urbana-Champaign, 2010. Available at http://www.math.uiuc.edu/∼ykim33/thesis.pdf.

[9] N. Naumann, M. Spitzweck, and P. A. Østvær. Motivic Landweber exactness. *Doc. Math.*, 14:551–593, 2009.

[10] B. Richter. An Atiyah-Hirzebruch spectral sequence for topological André-Quillen homology. *J. Pure Appl. Algebra*, 171(1):59–66, 2002.

[11] A. Robinson. Gamma homology, Lie representations and E_∞ multiplications. *Invent. Math.*, 152(2):331–348, 2003.

[12] A. Robinson, N. Naumann, M. Spitzweck, and P. A. Østvær. Robinson's E_∞-obstruction theory revisited. In preparation.

[13] O. Röndigs, M. Spitzweck, and P. A. Østvær. Motivic strict ring models for K-theory. *Proc. Amer. Math. Soc.*, 138(10):3509–3520, 2010.

[14] M. Spitzweck and P. A. Østvær. Motivic twisted K-theory. arXiv:1008.4915.

[15] M. Spitzweck and P. A. Østvær. The Bott inverted infinite projective space is homotopy algebraic K-theory. *Bull. Lond. Math. Soc.*, 41(2):281–292, 2009.

[16] V. Voevodsky. Open problems in the motivic stable homotopy theory. I. In *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 3–34. Int. Press, Somerville, MA, 2002.

Fakultät für Mathematik, Universität Regensburg, Germany.
e-mail: niko.naumann@mathematik.uni-regensburg.de

Department of Mathematics, University of Oslo, Norway.
e-mail: markussp@math.uio.no

Department of Mathematics, University of Oslo, Norway.
e-mail: paularne@math.uio.no

Existence and uniqueness of E_∞-structures on motivic K-theory spectra

Niko Naumann, Markus Spitzweck, Paul Arne Østvær

June 30, 2011

Abstract

We show the algebraic K-theory spectrum KGL, the motivic Adams summand ML and their connective covers have unique E_∞-structures, refining their naive multiplicative structures in the motivic stable homotopy category. These results are deduced from Γ-homology computations in motivic obstruction theory.

1 Introduction

Motivic homotopy theory intertwines classical algebraic geometry and modern algebraic topology. In this paper we study obstruction theory for E_∞-structures in the motivic setup. An E_∞-structure on a spectrum refers to a ring structure which is not just given up to homotopy, but where the homotopies encode a coherent homotopy commutative multiplication. Many of the examples of motivic ring spectra begin life as commutative monoids in the motivic stable homotopy category. We are interested in the following questions: When can the multiplicative structure of a given commutative monoid in the motivic stable homotopy category be refined to an E_∞-ring spectrum? And if such a refinement exists, when is it unique? The questions of existence and uniqueness of E_∞-structures and their many ramifications have been studied extensively in topology. The first motivic examples worked out in this paper are of K-theoretic interest.

The complex cobordism spectrum MU and its motivic analogue MGL have natural E_∞-structures. In the topological setup, Baker and Richter [1] have shown that the complex K-theory spectrum KU, the Adams summand L and the real K-theory spectrum KO admit unique E_∞-structures. The results in [1] are approached via the obstruction theory developed by Robinson in [12], where it is shown that existence and uniqueness of E_∞-structures are guaranteed provided certain Γ-cohomology groups vanish.

In our approach we rely on analogous results in the motivic setup. We show that the relevant motivic Γ-cohomology groups vanish in the case of the algebraic K-theory spectrum KGL (Theorem 4.6) and the motivic Adams summand ML introduced in this paper (see §6). The main ingredients in the proofs are new computations of the Γ-homology complexes of KU and L, see Theorem 4.3 and Lemma 6.3, and the Landweber base change formula for the motivic cooperations of KGL and ML. Our main result for KGL can be formulated as follows:

Theorem 1.1: *The algebraic K-theory spectrum KGL has a unique E_∞-structure refining its multiplication in the motivic stable homotopy category.*

The existence of the E_∞-structure on KGL was already known using the Bott inverted model for algebraic K-theory, see [13], [16], [3], but the analogous result for ML is new. The uniqueness part of the theorem is also new; it rules out the existence of any exotic E_∞-structures on KGL. We note that related motivic E_∞-structures have proven useful in the recent constructions of Atiyah-Hirzebruch types of spectral sequences for motivic twisted K-theory [15].

In topology, the Goerss-Hopkins-Miller obstruction theory [4] allows to gain control over *moduli spaces* of E_∞-structures. In favorable cases, such as for Lubin-Tate spectra, the moduli spaces are $K(\pi, 1)$'s giving rise to actions of certain automorphism groups as E_∞-maps. A motivic analogue of this obstruction theory seems to be within reach, but it has not been worked out.

In Section 5 we show that the connective cover kgl of the algebraic K-theory spectrum has a unique E_∞-structure, and ditto in Section 6 for the connective cover of the Adams summand. For the analogous topological results we refer to [2].

We conclude the introduction with an overview of the paper: In Section 2 we state the straightforward adaption of Robinson's obstruction theory to the motivic context, and point out its relevance in the proof of Theorem 1.1. In Section 3 we explain the consequences of our work for multiplicative structures on algebraic K-theory spectra. In Section 4 we show the basic input required for the obstruction theory is explicitly computable in case of algebraic K-theory, the main result being Theorem 4.6. Sections 5 and 6 discuss further examples to which we can successfully apply the obstruction theory, namely connective algebraic K-theory and the motivic analogue of the Adams summand.

2 Motivic obstruction theory

The aim of this section is to formulate a key result in motivic obstruction theory. It should be noted that a proof of Theorem 2.1 has not yet appeared in print, cf. [11].

To begin, fix a noetherian base scheme of finite Krull dimension with motivic stable homotopy category SH. (Modelled for example by the monoidal model category of motivic symmetric spectra developed by Jardine [8].)

Let E be a *commutative motivic ring spectrum*, i.e. a commutative and associative unitary monoid in $SH(S)$. Denote its coefficients by $R := \mathsf{E}_{**}$ and its cooperations by $\Lambda := \mathsf{E}_{**}\mathsf{E}$. We say E *satisfies the universal coefficient theorem* if for all $n \geq 1$ the Kronecker product yields an isomorphism

$$\mathsf{E}^{**}(\mathsf{E}^{\wedge n}) \xrightarrow{\cong} Hom_R(\Lambda^{\otimes_R n}, R).$$

Algebraic K-theory satisfies the universal coefficient theorem by [9, Theorem 9.3 (i)]. In this situation one can define trigraded motivic Γ-cohomology groups $\mathsf{H}\Gamma^*$ associated to R and Λ, cf. Section 4 for more details.

The almost identical motivic version of Robinson's result [12, Theorem 5.6] takes the following form.

Theorem 2.1: *Suppose* E *is a commutative motivic ring spectrum satisfying the universal coefficient theorem and the vanishing conditions* $\mathsf{H}\Gamma^{n,2-n,*}(\Lambda|R;R) = 0$ *for* $n \geq 4$ *and* $\mathsf{H}\Gamma^{n,1-n,*}(\Lambda|R;R) = 0$ *for* $n \geq 3$. *Then* E *admits an* E_∞*-structure unique up to homotopy.*

We note that our *proof of Theorem 1.1* is obtained by combining Theorem 2.1 for $\mathsf{E} = \mathsf{KGL}$ and Theorem 4.6.

3 Multiplicative structures on algebraic K-theory spectra

Let X be a scheme. The bipermutative structure on the category of coherent $\mathcal{O}_X$-modules gives rise to an E_∞-structure on the algebraic K-theory spectrum $K(X)$. One may ask if this is the only E_∞-structure refining its underlying homotopy commutative ring spectrum structure. It is known that for suitable finite Postnikov sections of the connective real K-theory spectrum ko, the analogous question has a negative answer, i.e. there *do* exist "exotic" E_∞-structures. We are unaware of any (classical) scheme X for which the answer to the above question is known, but one can show the following.

Theorem 3.1: *Let S be a separated and regular Noetherian scheme of finite Krull dimension. Assume*

$$\mathcal{K} : (Sm/S)^{op} \longrightarrow \{E_\infty - \text{ ring spectra}\}$$

is a presheaf of E_∞-ring spectra on the category of smooth S-schemes of finite type such that there is an equivalence $\Phi : \mathcal{K} \stackrel{\cong}{\to} K$ of commutative monoids in the homotopy category of presheaves of S^1-spectra. Then, for every X/S smooth, $\Phi(X)$ is an equivalence of E_∞-ring spectra.

Put informally, while we cannot rule out the existence of exotic multiplications on an individual algebraic K-theory spectrum, no such multiplications exist for the K-theory presheaf. Theorem 3.1 will be deduced from Theorem 1.1 in a forthcoming work by the second author [14]. In principle, one may approach this problem by studying, for a fixed scheme X, the Γ-cohomology of the extension

$$K_*(X) \to K(X)_*K(X).$$

However, it seems difficult to carry out such an analysis for non-empty schemes.

4 Algebraic K-theory KGL

In this section we shall present the Γ-cohomology computation showing there is a unique E_∞-structure on the algebraic K-theory spectrum KGL. Throughout we work over some noetherian base scheme of finite Krull dimension, which we omit from the notation.

There are two main ingredients which make this computation possible: First, the Γ-homology computation of $\mathsf{KU}_0\mathsf{KU}$ over $\mathsf{KU}_0 = \mathbf{Z}$, where KU is the complex K-theory spectrum. Second, we employ base change for the motivic cooperations of algebraic K-theory, as shown in our previous work [9].

4.1 The Γ-homology of $\mathsf{KU}_0\mathsf{KU}$ over KU_0

For a map $A \to B$ between commutative algebras we denote Robinson's Γ-homology complex by $\widetilde{\mathcal{K}}(B|A)$ [12, Definition 4.1]. Recall that $\widetilde{\mathcal{K}}(B|A)$ is a homological double complex of B-modules concentrated in the first quadrant. The same construction can be performed for maps between graded and bigraded algebras. In all cases we let $\mathcal{K}(B|A)$ denote the total complex associated with the double complex $\widetilde{\mathcal{K}}(B|A)$.

The Γ-cohomology

$$\mathsf{H}\Gamma^*(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0, -) = \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathsf{KU}_0\mathsf{KU}}(\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0), -)$$

has been computed for various coefficients in [1]. In what follows we require precise information about the complex $\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0)$ itself, since it satisfies a motivic base change property, cf. Lemma 4.4.

Lemma 4.1: *Let $X \in \mathsf{Ch}_{\geq 0}(\mathsf{Ab})$ be a non-negative chain complex of abelian groups. The following are equivalent:*

i) The canonical map $X \longrightarrow X \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{Q} = X \otimes_{\mathbf{Z}} \mathbf{Q}$ is a quasi isomorphism.

ii) For every prime p, there is a quasi isomorphism $X \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p \simeq 0$.

Proof. It is well known that X is formal [5, pg. 164], i.e. there is a quasi-isomorphism

$$X \simeq \bigoplus_{n\geq 0} H_n(X)[n].$$

(For an abelian group A and integer n, we let $A[n]$ denote the chain complex that consists of A concentrated in degree n.) Hence for every prime p,

$$X \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p \simeq \bigoplus_{n\geq 0} \left(H_n(X)[n] \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p\right).$$

By resolving $\mathbf{F}_p = (\mathbf{Z} \xrightarrow{\cdot p} \mathbf{Z})$ one finds an isomorphism

$$H_*(A[n] \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p) \cong (A/pA)[n] \oplus A\{p\}[n+1]$$

for every abelian group A and integer n. Here $A\{p\}$ is shorthand for $\{x \in A \,|\, px = 0\}$. In summary, ii) holds if and only if the multiplication by p map

$$\cdot p : H_*(X) \longrightarrow H_*(X)$$

is an isomorphism for every prime p. The latter is equivalent to i). $\square$

We shall use the previous lemma in order to study cotangent complexes introduced by Illusie in [7]. Let R be a ring and set $R_{\mathbf{Q}} := R \otimes_{\mathbf{Z}} \mathbf{Q}$. Then there is a canonical map

$$\tau_R : \mathbb{L}_{R/\mathbf{Z}} \longrightarrow \mathbb{L}_{R/\mathbf{Z}} \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{Q} \simeq \mathbb{L}_{R/\mathbf{Z}} \otimes^{\mathbf{L}}_{R} R_{\mathbf{Q}} \xrightarrow{\simeq} \mathbb{L}_{R_{\mathbf{Q}}/\mathbf{Q}}$$

of cotangent complexes in $\mathsf{Ho}(\mathsf{Ch}_{\geq 0}(\mathbf{Z}))$. The first quasi isomorphism is obvious, while the second one is an instance of flat base change for cotangent complexes.

Lemma 4.2: *The following are equivalent:*

i) τ_R *is a quasi isomorphism.*

ii) For every prime p*, there is a quasi isomorphism* $\mathbb{L}_{R/\mathbf{Z}} \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p \simeq 0$.

If the abelian group underlying R *is torsion free, then i) and ii) are equivalent to*

iii) For every prime p, $\mathbb{L}_{(R/pR)/\mathbf{F}_p} \simeq 0$.

Proof. The equivalence of i) and ii) follows by applying Lemma 4.1 to $X = \mathbb{L}_{R/\mathbf{Z}}$. If R is torsion free, then it is flat as a $\mathbf{Z}$-algebra. Hence, by flat base change, there exists a quasi isomorphism

$$\mathbb{L}_{R/\mathbf{Z}} \otimes^{\mathbf{L}}_{\mathbf{Z}} \mathbf{F}_p \simeq \mathbb{L}_{(R/pR)/\mathbf{F}_p}.$$

□

The following is our analogue for Robinson's Γ-homology complex of the Baker-Richter result [1, Theorem 5.1].

Theorem 4.3: *i) Let* R *be a torsion free ring such that* $\mathbb{L}_{(R/pR)/\mathbf{F}_p} \simeq 0$ *for every prime* p*, e.g. assume that* $\mathbf{F}_p \to R/pR$ *is ind-étale for all* p*. Then there is a quasi isomorphism*

$$\mathcal{K}(R|\mathbf{Z}) \simeq \mathcal{K}(R_{\mathbf{Q}}|\mathbf{Q})$$

in the derived category of R*-modules.*

ii) There is a quasi isomorphism

$$\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0) \simeq (\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}}[0]$$

in the derived category of $\mathsf{KU}_0\mathsf{KU}$*-modules.*

Proof. i) The Atiyah-Hirzebruch spectral sequence noted in [10, Remark 2.3] takes the form

$$E^2_{p,q} = H^p(\mathbb{L}_{R/\mathbf{Z}} \otimes^{\mathbf{L}}_{\mathbf{Z}} \Gamma^q(\mathbf{Z}[x]|\mathbf{Z})) \Rightarrow H^{p+q}(\mathcal{K}(R|\mathbf{Z})).$$

Our assumptions on R and Lemma 4.2 imply that the E^2-page is comprised of $\mathbf{Q}$-vector spaces. Hence so is the abutment, and there exists a quasi isomorphism between complexes of R-modules

$$\mathcal{K}(R|\mathbf{Z}) \xrightarrow{\simeq} \mathcal{K}(R|\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Moreover, by Lemma 4.4, there is a quasi isomorphism

$$\mathcal{K}(R|\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{Q} \simeq \mathcal{K}(R_{\mathbf{Q}}|\mathbf{Q}).$$

ii) According to [1, Theorem 3.1, Corollary 3.4, (a)] and the Hopf algebra isomorphism $A^{st} \simeq \mathsf{KU}_0\mathsf{KU}$ [1, Proposition 6.1], the ring $R := \mathsf{KU}_0\mathsf{KU}$ satisfies the assumptions of part i)[1]. Now since $\mathsf{KU}_0 \cong \mathbf{Z}$,

$$\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0) \simeq \mathcal{K}((\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}}|\mathbf{Q}).$$

We have that $(\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}} \simeq \mathbf{Q}[w^{\pm 1}]$ [1, Theorem 3.2, (c)] is a smooth $\mathbf{Q}$-algebra. Hence, since Γ-cohomology agrees with André-Quillen cohomology over $\mathbf{Q}$, there are quasi isomorphisms

$$\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0) \simeq \Omega^1_{\mathbf{Q}[w^{\pm 1}]|\mathbf{Q}}[0] \simeq (\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}}[0].$$

□

4.2 The Γ-homology of $\mathsf{KGL}_{**}\mathsf{KGL}$ over KGL_{**}

The strategy in what follows is to combine the computations for KU in §4.1 with motivic Landweber exactness [9]. To this end we require the following general base change result, which was also used in the proof of Theorem 4.3.

Lemma 4.4: *For a pushout of ordinary, graded or bigraded commutative algebras*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

there are isomorphisms between complexes of D-modules

$$\mathcal{K}(D|C) \cong \mathcal{K}(B|A) \otimes_B D \cong \mathcal{K}(B|A) \otimes_A C.$$

If B is flat over A, then $\widetilde{\mathcal{K}}(B|A)$ is a first quadrant homological double complex of flat B-modules; thus, in the derived category of D-modules there are quasi isomorphisms

$$\mathcal{K}(D|C) \simeq \mathcal{K}(B|A) \otimes^{\mathbf{L}}_B D \simeq \mathcal{K}(B|A) \otimes^{\mathbf{L}}_A C.$$

Proof. Following the notation in [12, §4], let $(B|A)^{\otimes}$ denote the tensor algebra of B over A. Then $(B|A)^{\otimes} \otimes_A B$ has a natural Γ-module structure over B, cf. [12, §4]. Here Γ denotes the category of finite based sets and basepoint preserving maps. It follows

[1] This also follows easily from Landweber exactness of KU.

that $((B|A)^{\otimes} \otimes_A B) \otimes_B D$ is a Γ-module over D. Moreover, by base change for tensor algebras, there exists an isomorphism of Γ-modules in D-modules

$$((B|A)^{\otimes} \otimes_A B) \otimes_B D \cong (D|C)^{\otimes} \otimes_C D.$$

Here we use that the Γ-module structure on $(B|A)^{\otimes} \otimes_A M$, for M a B-module, is given as follows: For a map $\varphi\colon [m] \to [n]$ between finite pointed sets,

$$(B \otimes_A B \otimes_A \cdots \otimes_A B) \otimes_A M \to (B \otimes_A B \otimes_A \cdots \otimes_A B) \otimes_A M$$

sends $b_1 \otimes \cdots \otimes b_m \otimes m$ to

$$(\prod_{i\in\varphi^{-1}(1)} b_i) \otimes \cdots \otimes (\prod_{i\in\varphi^{-1}(n)} b_i) \otimes ((\prod_{i\in\varphi^{-1}(0)} b_i) \cdot m).$$

By convention, if $\varphi^{-1}(j) = \emptyset$ then $\prod_{i\in\varphi^{-1}(j)} b_i = 1$. Robinson's Ξ-construction yields an isomorphism between double complexes of D-modules

$$\widetilde{\mathcal{K}}(D|C) = \Xi((D|C)^{\otimes} \otimes_C D) \cong \Xi(((B|A)^{\otimes} \otimes_A B) \otimes_B D).$$

Inspection of the Ξ-construction reveals there is an isomorphism

$$\Xi(((B|A)^{\otimes} \otimes_A B) \otimes_B D) \cong \Xi((B|A)^{\otimes} \otimes_A B) \otimes_B D.$$

By definition, this double complex of D-modules is $\widetilde{\mathcal{K}}(B|A) \otimes_B D \cong \widetilde{\mathcal{K}}(B|A) \otimes_A C$. This proves the first assertion by comparing the corresponding total complexes. The remaining claims follow easily. $\square$

Next we recall the structure of the motivic cooperations of the algebraic K-theory spectrum KGL. The algebras we shall consider are bigraded as follows: $\mathsf{KU}_0 \cong \mathbf{Z}$ in bidegree $(0,0)$ and $\mathsf{KU}_* \cong \mathbf{Z}[\beta^{\pm 1}]$ with the Bott-element β in bidegree $(2,1)$. With these conventions, there is a canonical bigraded map

$$\mathsf{KU}_* \to \mathsf{KGL}_{**}.$$

Lemma 4.5: *There are pushouts of bigraded algebras*

$$\begin{tikzcd} \mathsf{KU}_* \arrow[r, "\eta_L"] \arrow[d] & \mathsf{KU}_*\mathsf{KU} \arrow[d] \\ \mathsf{KGL}_{**} \arrow[r, "\eta_L"] & \mathsf{KGL}_{**}\mathsf{KGL} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{KU}_0 \arrow[r, "(\eta_L)_0"] \arrow[d] & \mathsf{KU}_0\mathsf{KU} \arrow[d] \\ \mathsf{KU}_* \arrow[r, "\eta_L"] & \mathsf{KU}_*\mathsf{KU} \end{tikzcd}$$

and a quasi isomorphism in the derived category of $\mathsf{KGL}_{**}\mathsf{KGL}$*-modules*

$$\mathcal{K}(\mathsf{KGL}_{**}\mathsf{KGL}|\mathsf{KGL}_{**}) \simeq \mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathsf{KU}_0) \otimes^{\mathbf{L}}_{\mathsf{KU}_0\mathsf{KU}} \mathsf{KGL}_{**}\mathsf{KGL}.$$

Proof. Here, η_L is a generic notation for the left unit of some flat Hopf-algebroid. The first pushout is shown in [9, Proposition 9.1, (c)]. The second pushout is in [1]. Applying Lemma 4.4 twice gives the claimed quasi isomorphism. $\square$

Next we compute the Γ-cohomology of the motivic cooperations of KGL.

Theorem 4.6: *i) There is an isomorphism*

$$\mathsf{H}\Gamma^{*,*,*}(\mathsf{KGL}_{**}\mathsf{KGL}|\mathsf{KGL}_{**};\mathsf{KGL}_{**}) \cong \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathbf{Z}}(\mathbf{Q}[0],\mathsf{KGL}_{**}).$$

ii) For all $s \geq 2$,

$$\mathsf{H}\Gamma^{s,*,*}(\mathsf{KGL}_{**}\mathsf{KGL}|\mathsf{KGL}_{**};\mathsf{KGL}_{**}) = 0.$$

Proof. i) By the definition of Γ-cohomology and the results in this Subsection there are isomorphisms

$$\begin{array}{rl} & \mathsf{H}\Gamma^{*,*,*}(\mathsf{KGL}_{**}\mathsf{KGL}|\mathsf{KGL}_{**};\mathsf{KGL}_{**}) \\ = & \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathsf{KGL}_{**}\mathsf{KGL}}(\mathcal{K}(\mathsf{KGL}_{**}\mathsf{KGL}|\mathsf{KGL}_{**}),\mathsf{KGL}_{**}) \\ \cong & \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathsf{KGL}_{**}\mathsf{KGL}}(\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathbf{Z}) \otimes^{\mathbf{L}}_{\mathsf{KU}_0\mathsf{KU}} \mathsf{KGL}_{**}\mathsf{KGL},\mathsf{KGL}_{**}) \\ \cong & \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathsf{KU}_0\mathsf{KU}}(\mathcal{K}(\mathsf{KU}_0\mathsf{KU}|\mathbf{Z}),\mathsf{KGL}_{**}) \\ \cong & \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathsf{KU}_0\mathsf{KU}}((\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}}[0],\mathsf{KGL}_{**}) \\ \cong & \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathbf{Z}}(\mathbf{Q}[0],\mathsf{KGL}_{**}). \end{array}$$

ii) This follows from i) since $\mathbf{Z}$ has global dimension 1.

$\square$

Remark 4.7: *It is an exercise to compute $\mathbf{R}\mathsf{Hom}_{\mathbf{Z}}(\mathbf{Q},-)$ applied to finitely generated abelian groups. This explicates our Gamma-cohomology computation in cohomological degrees 0 and 1 for base schemes with finitely generated algebraic K-groups, e.g. finite fields and number rings. The computation $\mathbf{R}\mathsf{Hom}_{\mathbf{Z}}(\mathbf{Q},\mathbf{Z}) \simeq \hat{\mathbf{Z}}/\mathbf{Z}[1]$ shows our results imply [1, Corollary 5.2].*

5 Connective algebraic K-theory kgl

We define the connective algebraic K-theory spectrum kgl as the effective part $\mathsf{f}_0\mathsf{KGL}$ of KGL. Recall that the functor f_i defined in [17] projects from the motivic stable homotopy category to its ith effective part. Note that $\mathsf{f}_0\mathsf{KGL}$ is a commutative monoid in the motivic stable homotopy category since projection to the effective part is a lax symmetric

monoidal functor (because it is right adjoint to a monoidal functor). For $i \in \mathbf{Z}$ there exists a natural map $\mathsf{f}_{i+1}\mathsf{KGL} \to \mathsf{f}_i\mathsf{KGL}$ in the motivic stable homotopy category with cofiber the ith slice of KGL. With these definitions, $\mathsf{KGL} \cong \operatorname{hocolim} \mathsf{f}_i\mathsf{KGL}$ (this is true for any motivic spectrum, cf. [17, Lemma 4.2]). Bott periodicity for algebraic K-theory implies that $\mathsf{f}_{i+1}\mathsf{KGL} \cong \Sigma^{2,1}\mathsf{f}_i\mathsf{KGL}$. This allows to recast the colimit as $\operatorname{hocolim} \Sigma^{2i,i}\mathsf{kgl}$ with multiplication by the Bott element β in $\mathsf{kgl}^{-2,-1} \cong \mathsf{KGL}^{-2,-1}$ as the transition map at each stage. We summarize these observations in a lemma.

Lemma 5.1: *The algebraic K-theory spectrum KGL is isomorphic in the motivic stable homotopy category to the Bott inverted connective algebraic K-theory spectrum $\mathsf{kgl}[\beta^{-1}]$.*

Theorem 5.2: *The connective algebraic K-theory spectrum kgl has a unique E_∞-structure refining its multiplication in the motivic stable homotopy category.*

Proof. The connective cover functor f_0 preserves E_∞-structures [6]. Thus the existence of an E_∞-structure on kgl is ensured. We note that inverting the Bott element can be refined to the level of motivic E_∞-ring spectra by the methods employed in [13]. Thus, by Lemma 5.1, starting out with any two E_∞-structures on kgl produces two E_∞-structures on KGL, which coincide by the uniqueness result for E_∞-structures on KGL . Applying f_0 recovers the two given E_∞-structures on kgl: If X is E_∞ with $\varphi\colon X \simeq \mathsf{kgl}$ as ring spectra, then there is a canonical E_∞-map $X \to X[\beta'^{-1}]$, where β' is the image of the Bott element under φ. Since X is an effective motivic spectrum, this map factors as an E_∞-map $X \to \mathsf{f}_0(X[\beta'^{-1}])$. By construction of kgl the latter map is an equivalence. This shows the two given E_∞-structures on kgl coincide. $\square$

6 The motivic Adams summands ML and ml

Let BP denote the Brown-Peterson spectrum for a fixed prime number p. Then the coefficient ring $\mathsf{KU}_{(p)*}$ of the p-localized complex K-theory spectrum is a BP_*-module via the ring map $\mathsf{BP}_* \to \mathsf{MU}_{(p)*}$ which classifies the p-typicalization of the formal group law over $\mathsf{MU}_{(p)*}$. The $\mathsf{MU}_{(p)*}$-algebra structure on $\mathsf{KU}_{(p)*}$ is induced from the natural orientation $\mathsf{MU} \to \mathsf{KU}$. With this BP_*-module structure, $\mathsf{KU}_{(p)*}$ splits into a direct sum of the $\Sigma^{2i}\mathsf{L}_*$ for $0 \leq i \leq p-2$, where L is the Adams summand of $\mathsf{KU}_{(p)}$. Thus motivic Landweber exactness [9] over the motivic Brown-Peterson spectrum MBP produces a splitting of motivic spectra

$$\mathsf{KGL}_{(p)} = \bigvee_{i=0}^{p-2} \Sigma^{2i,i}\mathsf{ML}.$$

We refer to ML as the motivic Adams summand of algebraic K-theory.

Since L_* is an BP_*-algebra and there are no nontrivial phantom maps from any smash power of ML to ML, which follows from [9, Remark 9.8, (ii)] since ML is a retract of $\mathsf{KGL}_{(p)}$, we deduce that the corresponding ring homology theory induces a commutative monoid structure on ML in the motivic stable homotopy category.

We define the connective motivic Adams summand ml to be $f_0\mathsf{ML}$. It is also a commutative monoid in the motivic homotopy category.

Theorem 6.1: *The motivic Adams summand* ML *has a unique* E_∞*-structure refining its multiplication in the motivic stable homotopy category. The same result holds for the connective motivic Adams summand* ml.

The construction of ML as a motivic Landweber exact spectrum makes the following result evident on account of the proof of Lemma 4.5.

Lemma 6.2: *There exist pushout squares of bigraded algebras*

$$\begin{array}{ccc} \mathsf{L}_* & \xrightarrow{\eta_L} & \mathsf{L}_*\mathsf{L} \\ \downarrow & & \downarrow \\ \mathsf{ML}_{**} & \xrightarrow{\eta_L} & \mathsf{ML}_{**}\mathsf{ML} \end{array} \qquad \begin{array}{ccc} \mathsf{L}_0 & \xrightarrow{(\eta_L)_0} & \mathsf{L}_0\mathsf{L} \\ \downarrow & & \downarrow \\ \mathsf{L}_* & \xrightarrow{\eta_L} & \mathsf{L}_*\mathsf{L} \end{array}$$

and a quasi isomorphism in the derived category of $\mathsf{ML}_{**}\mathsf{ML}$*-modules*

$$\mathcal{K}(\mathsf{ML}_{**}\mathsf{ML}|\mathsf{ML}_{**}) \simeq \mathcal{K}(\mathsf{L}_0\mathsf{L}|\mathsf{L}_0) \otimes^{\mathbf{L}}_{\mathsf{L}_0\mathsf{L}} \mathsf{ML}_{**}\mathsf{ML}.$$

Next we show the analog of Theorem 4.3, ii) for the motivic Adams summand.

Lemma 6.3: *In the derived category of* $\mathsf{L}_0\mathsf{L}$*-modules, there is a quasi isomorphism*

$$\mathcal{K}(\mathsf{L}_0\mathsf{L}|\mathsf{L}_0) \simeq (\mathsf{L}_0\mathsf{L})_{\mathbf{Q}}[0].$$

Proof. In the notation of [1, Proposition 6.1] there is an isomorphism between Hopf algebras $\mathsf{L}_0\mathsf{L} \cong {}^{\zeta}A^{st}_{(p)}$. Recall that ${}^{\zeta}A^{st}_{(p)}$ is a free $\mathbf{Z}_{(p)}$-module on a countable basis and ${}^{\zeta}A^{st}_{(p)}/p{}^{\zeta}A^{st}_{(p)}$ is a formally étale $\mathbf{F}_p$-algebra [1, Theorem 3.3(c), Corollary 4.2]. Applying Theorem 4.3, i) to $R = \mathsf{L}_0\mathsf{L}$ and using that $(\mathsf{L}_0\mathsf{L})_{\mathbf{Q}} \simeq \mathbf{Q}[v^{\pm 1}]$ by Landweber exactness, where $v = w^{p-1}$ and $(\mathsf{KU}_0\mathsf{KU})_{\mathbf{Q}} \cong \mathbf{Q}[w^{\pm 1}]$, we find

$$\mathcal{K}(\mathsf{L}_0\mathsf{L}|\mathsf{L}_0) \simeq \Omega^1_{\mathbf{Q}[v^{\pm 1}]|\mathbf{Q}}[0] \simeq (\mathsf{L}_0\mathsf{L})_{\mathbf{Q}}[0].$$

□

Lemmas 6.2 and 6.3 imply there is a quasi isomorphism

$$\mathsf{H}\Gamma^{*,*,*}(\mathsf{ML}_{**}\mathsf{ML}|\mathsf{ML}_{**};\mathsf{ML}_{**}) \simeq \mathbf{H}^*\mathbf{R}\mathsf{Hom}_{\mathbf{Z}}(\mathbf{Q}[0],\mathsf{ML}_{**}).$$

Thus the part of Theorem 6.1 dealing with ML follows, since for all $s \geq 2$,

$$\mathsf{H}\Gamma^{s,*,*}(\mathsf{ML}_{**}\mathsf{ML}|\mathsf{ML}_{**};\mathsf{ML}_{**}) = 0. \tag{1}$$

The assertion about ml follows by the exact same type of argument as for kgl. The periodicity operator in this case is $v_1 \in \mathsf{ml}^{2(1-p),1-p} = \mathsf{ML}^{2(1-p),1-p}$.

Acknowledgements. The main result Theorem 1.1 of this paper was announced by the first named author at the 2009 Münster workshop on Motivic Homotopy Theory. He thanks the organizers E. M. Friedlander, G. Quick and P. A. Østvær for the invitation, and P. A. Østvær for hospitality while visiting the University of Oslo, where the major part of this work was finalized. The authors thank A. Robinson for discussions and an anonymous referee for an insightful report which led to an improvement of the exposition.

References

[1] A. Baker and B. Richter. On the Γ-cohomology of rings of numerical polynomials and E_∞ structures on K-theory. *Comment. Math. Helv.*, 80(4):691–723, 2005.

[2] A. Baker and B. Richter. Uniqueness of E_∞ structures for connective covers. *Proc. Amer. Math. Soc.*, 136(2):707–714 (electronic), 2008.

[3] D. Gepner and V. Snaith. On the motivic spectra representing algebraic cobordism and algebraic K-theory. *Doc. Math.*, 14:359–396, 2009.

[4] P. G. Goerss and M. J. Hopkins. Moduli spaces of commutative ring spectra. In *Structured ring spectra*, volume 315 of *London Math. Soc. Lecture Note Ser.*, pages 151–200. Cambridge Univ. Press, Cambridge, 2004.

[5] P. G. Goerss and J. F. Jardine. *Simplicial homotopy theory*, volume 174 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1999.

[6] J. J. Gutiérrez, O. Röndigs, M. Spitzweck, and P. A. Østvær. Slices and colored operads. arXiv:1012.3301.

[7] L. Illusie. *Complexe cotangent et déformations. I.* Lecture Notes in Mathematics, Vol. 239. Springer-Verlag, Berlin, 1971.

[8] J. F. Jardine. Motivic symmetric spectra. *Doc. Math.*, 5:445–553 (electronic), 2000.

[9] N. Naumann, M. Spitzweck, and P. A. Østvær. Motivic Landweber exactness. *Doc. Math.*, 14:551–593, 2009.

[10] B. Richter. An Atiyah-Hirzebruch spectral sequence for topological André-Quillen homology. *J. Pure Appl. Algebra*, 171(1):59–66, 2002.

[11] A. Robinson. On E_∞-obstruction theory. In preparation.

[12] A. Robinson. Gamma homology, Lie representations and E_∞ multiplications. *Invent. Math.*, 152(2):331–348, 2003.

[13] O. Röndigs, M. Spitzweck, and P. A. Østvær. Motivic strict ring models for K-theory. *Proc. Amer. Math. Soc.*, 138(10):3509–3520, 2010.

[14] M Spitzweck. In preparation.

[15] M. Spitzweck and P. A. Østvær. Motivic twisted K-theory. arXiv:1008.4915.

[16] M. Spitzweck and P. A. Østvær. The Bott inverted infinite projective space is homotopy algebraic K-theory. *Bull. Lond. Math. Soc.*, 41(2):281–292, 2009.

[17] V. Voevodsky. Open problems in the motivic stable homotopy theory. I. In *Motives, polylogarithms and Hodge theory, Part I (Irvine, CA, 1998)*, volume 3 of *Int. Press Lect. Ser.*, pages 3–34. Int. Press, Somerville, MA, 2002.

Fakultät für Mathematik, Universität Regensburg, Germany.
e-mail: niko.naumann@mathematik.uni-regensburg.de

Fakultät für Mathematik, Universität Regensburg, Germany.
e-mail: Markus.Spitzweck@mathematik.uni-regensburg.de

Department of Mathematics, University of Oslo, Norway.
e-mail: paularne@math.uio.no